Polynomials in Isabelle for the Working Mathematician

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`hg clone https://hg.risc.uni-linz.ac.at/wneuper/poly`
Isabelle download `http://isabelle.in.tum.de` .

Slides `https://hg.risc.uni-linz.ac.at/wneuper/poly-demo.pdf`

Outline

1. Joint work Computer Algebra — Isabelle
2. Towards a Polynomial Package in Isabelle
   - An abstract polynomial for proofs
   - Proofs and algorithms within one system
   - Polynomial representations: distributive – recursive
   - Automated code generation preserves logical properties
3. Demo: the proof-of-concept prototype
   - Isabelle/jEdit: towards a prover IDE
   - Isabelle's mechanisms "typedef" and "instantiation"
   - Definitions – proofs – algorithms – code generation
   - Questions . . .
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Joint work

Aims	**Computer Algebra**	**Isabelle development**
For the working mathematician	*n.n.* *n.n.* design definitions prove theorems verify algorithms	
develop a polynomial package	*Wolfgang Schreiner* *RISC Linz* SAGE: basic operations representations proof: representations $\equiv$ proof: polynomial ring, . . .	*Andreas Lochbihler* *ETHZ Zürich* basic operations representations proof: representations $\equiv$ proof: polynomial ring, . . .
for efficient and verified algorithms.		*Florian Haftmann* *TU München* from Isabelle definitions generate code (Scala, Haskell, SML, . . .) preserving verification

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Abstract polynomial

In Franz Winkler, Polynomial Algorithms, p.17:

Definition

An *n-variate polynomial over* the ring R is a mapping $p : \mathcal{N}_0^n \to R$, $(i_1, \ldots, i_n) \mapsto p_{1_i}, \ldots, p_{1_n}$ such that $p_{1_i}, \ldots, p_{1_n} = 0$ nearly everywhere. (Notation $p = \Sigma p_{1_i}, \ldots, p_{1_n} \cdot x^{i_1} \cdots x^{i_n}$)

In Isabelle's prototype[1]:

```
typedef ('a, 'b) poly_mapping =
  "{f :: 'a => 'b::zero. finite {x. f x ≠ 0}}"

typedef 'a mpoly =
  "UNIV::((nat,nat) poly_mapping, 'a::zero) poly_mapping
```

---

[1] https://hg.risc.uni-linz.ac.at/wneuper/poly/file/28e5ebbe5db5/Poly_Mapping.thy

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Proofs and algorithms

A lemma and a (automated) proof

```
lemma gcd_mod:  "gcd (q * b + r) (b::int) = gcd b r"
  by (metis gcd_commute_int gcd_red_int
    mod_mult_self1 add_commute)
```

and an algorithm (written using Isabelle’s “function package”)

```
function euclid ::  "'a::ring_div => 'a => 'a"
  where "euclid a b =
    (if b = 0 then a else euclid b (a mod b))"
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and a (omitted) proof about the algorithm

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theorem euclid_gcd:  "euclid (a::ring_div) b = gcd a b"
  proof sorry
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all within one system, within Isabelle.

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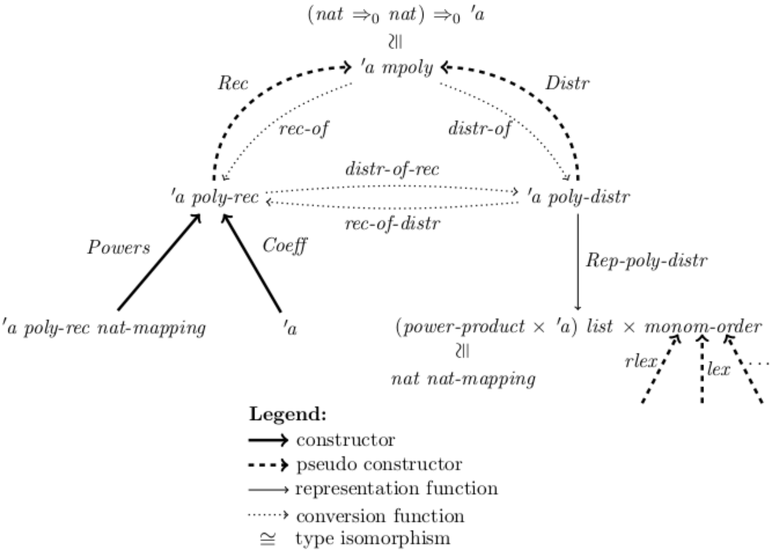
Polynomial representations



$(nat \Rightarrow_0 nat) \Rightarrow_0 {'a}$

$\cong$

'a mpoly

Rec

Distr

rec-of

distr-of

distr-of-rec

'a poly-rec

'a poly-distr

rec-of-distr

Powers

Coeff

Rep-poly-distr

'a poly-rec nat-mapping

'a

(power-product × 'a) list × monom-order

$\cong$

nat nat-mapping

rlex

lex

. . .

Legend:

$\longrightarrow$ constructor

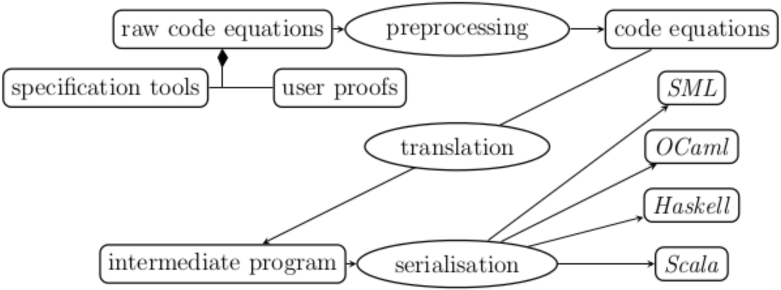
- - -> pseudo constructor

$\longrightarrow$ representation function

······> conversion function

$\cong$ type isomorphism

Automated code generation



raw code equations

preprocessing

code equations

specification tools

user proofs

translation

intermediate program

serialisation

SML

OCaml

Haskell

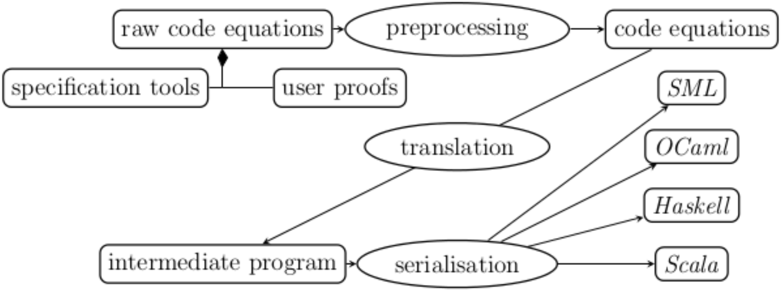
Scala

"code equations" are equational theorems in Isabelle/HOL.

Partial correctness of theorems transfers to generated programs: rewrite steps in the program can be simulated in the logic.

The generator's components can be customized individually.

Automated code generation



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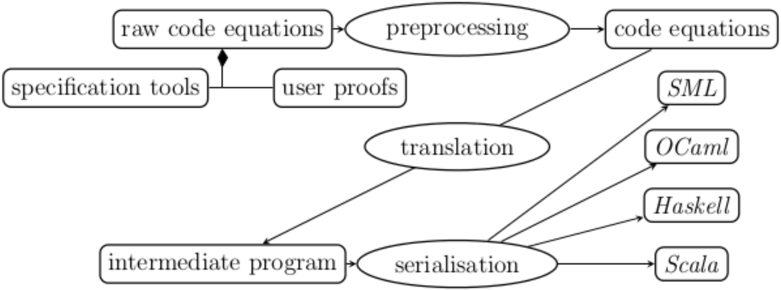

raw code equations
preprocessing
code equations
specification tools
user proofs
translation
intermediate program
serialisation
SML
OCaml
Haskell
Scala

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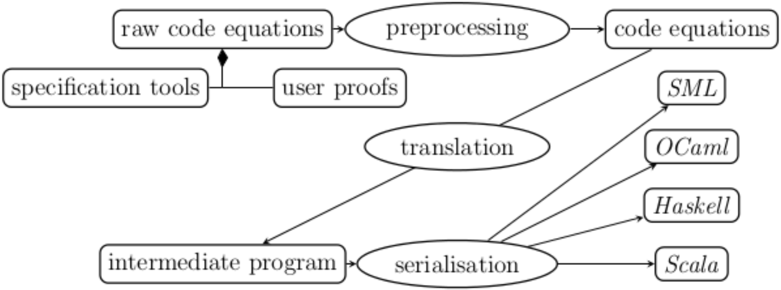

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Isabelle/jEdit

see Poly_Demo.thy

“typedef” and “instantiation”

see Poly_Demo.thy

Definitions – proofs – . . .

see Poly_Demo.thy

Questions

see Poly_Demo.thy

Thank you for attention !

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Short introduction to the prototype package:

`https://hg.risc.uni-linz.ac.at/wneuper/polyintro.pdf`