# Logic and Proving

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1. **The Language of Logic**
2. The Art of Proving
3. The RISC ProofNavigator



## The Language of Logic

Two kinds of syntactic phrases.

- Term $T$ denoting an object.
  - Variable $x$
  - Object constant $c$
  - Function application $f(T_1, \dots, T_n)$
    $n$-ary function constant $f$ (may be written infix)
- Formula $F$ denoting a truth value.
  - Atomic formula $p(T_1, \dots, T_n)$ (may be written infix)
    $n$-ary predicate constant $p$.
  - Negation $\neg F$ ("not $F$")
  - Conjunction $F_1 \wedge F_2$ ("$F_1$ and $F_2$")
  - Disjunction $F_1 \vee F_2$ ("$F_1$ or $F_2$")
  - Implication $F_1 \Rightarrow F_2$ ("if $F_1$, then $F_2$")
  - Equivalence $F_1 \Leftrightarrow F_2$ ("if $F_1$, then $F_2$, and vice versa")
  - Universal quantification $\forall x : F$ ("for all $x$, $F$")
  - Existential quantification $\exists x : F$ ( "for some $x$, $F$")



## Syntactic Shortcuts

- $\forall x_1, \dots, x_n : F$
  - $\forall x_1 : \dots : \forall x_n : F$
- $\exists x_1, \dots, x_n : F$
  - $\exists x_1 : \dots : \exists x_n : F$
- $\forall x \in S : F$
  - $\forall x : x \in S \Rightarrow F$
- $\exists x \in S : F$
  - $\exists x : x \in S \wedge F$

Help to make formulas more readable.

## Examples

Terms and formulas may appear in various syntactic forms.

- Terms:

$$\begin{array}{l} \exp(x) \\ a \cdot b + 1 \\ a[i] \cdot b \\ \sqrt{\frac{x^2+2x+1}{(y+1)^2}} \end{array}$$

- Formulas:

$$\begin{array}{l} a^2 + b^2 = c^2 \\ n \mid 2n \\ \forall x \in \mathbb{N} : x \geq 0 \\ \forall x \in \mathbb{N} : 2|x \vee 2|(x+1) \\ \forall x \in \mathbb{N}, y \in \mathbb{N} : x < y \Rightarrow \\ \qquad \exists z \in \mathbb{N} : x + z = y \end{array}$$

Terms and formulas may be nested arbitrarily deeply.



## The Meaning of Formulas

- Atomic formula $p(T_1, \ldots, T_n)$
  - True if the predicate denoted by $p$ holds for the values of $T_1, \ldots, T_n$.
- Negation $\neg F$
  - True if and only if $F$ is false.
- Conjunction $F_1 \wedge F_2$ ("$F_1$ and $F_2$")
  - True if and only if $F_1$ and $F_2$ are both true.
- Disjunction $F_1 \vee F_2$ ("$F_1$ or $F_2$")
  - True if and only if at least one of $F_1$ or $F_2$ is true.
- Implication $F_1 \Rightarrow F_2$ ("if $F_1$, then $F_2$")
  - False if and only if $F_1$ is true and $F_2$ is false.
- Equivalence $F_1 \Leftrightarrow F_2$ ("if $F_1$, then $F_2$, and vice versa")
  - True if and only if $F_1$ and $F_2$ are both true or both false.
- Universal quantification $\forall x : F$ ("for all $x$, $F$")
  - True if and only if $F$ is true for every possible value assignment of $x$.
- Existential quantification $\exists x : F$ ("for some $x$, $F$")
  - True if and only if $F$ is true for at least one value assignment of $x$.



## Example

We assume the domain of natural numbers and the "classical" interpretation of constants 1, 2, +, =, <.

- $1 + 1 = 2$
  - True.
- $1 + 1 = 2 \vee 2 + 2 = 2$
  - True.
- $1 + 1 = 2 \wedge 2 + 2 = 2$
  - False.
- $1 + 1 = 2 \Rightarrow 2 = 1 + 1$
  - True.
- $1 + 1 = 1 \Rightarrow 2 + 2 = 2$
  - True.
- $1 + 1 = 2 \Rightarrow 2 + 2 = 2$
  - False.
- $1 + 1 = 1 \Leftrightarrow 2 + 2 = 2$
  - True.



## Example

- $x + 1 = 1 + x$
  - True, for every assignment of a number $a$ to variable $x$.
- $\forall x : x + 1 = 1 + x$
  - True (because for every assignment $a$ to $x$, $x + 1 = 1 + x$ is true).
- $x + 1 = 2$
  - If $x$ is assigned "one", the formula is true.
  - If $x$ is assigned "two", the formula is false.
- $\exists x : x + 1 = 2$
  - True (because $x + 1 = 2$ is true for assignment "one" to $x$).
- $\forall x : x + 1 = 2$
  - False (because $x + 1 = 2$ is false for assignment "two" to $x$).
- $\forall x : \exists y : x < y$
  - True (because for every assignment $a$ to $x$, there exists the assignment $a + 1$ to $y$ which makes $x < y$ true).
- $\exists y : \forall x : x < y$
  - False (because for every assignment $a$ to $y$, there is the assignment $a + 1$ to $x$ which makes $x < y$ false).

## Formula Equivalences

Formulas may be replaced by equivalent formulas.

- $\neg\neg F_1 \leftrightsquigarrow F_1$
- $\neg(F_1 \wedge F_2) \leftrightsquigarrow \neg F_1 \vee \neg F_2$
- $\neg(F_1 \vee F_2) \leftrightsquigarrow \neg F_1 \wedge \neg F_2$
- $\neg(F_1 \Rightarrow F_2) \leftrightsquigarrow F_1 \wedge \neg F_2$
- $\neg\forall x : F \leftrightsquigarrow \exists x : \neg F$
- $\neg\exists x : F \leftrightsquigarrow \forall x : \neg F$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_2 \Rightarrow \neg F_1$
- $F_1 \Rightarrow F_2 \leftrightsquigarrow \neg F_1 \vee F_2$
- $F_1 \Leftrightarrow F_2 \leftrightsquigarrow \neg F_1 \Leftrightarrow \neg F_2$
- ...

Familiarity with manipulation of formulas is important.



## Example

- "All swans are white or black."
  - $\forall x : swan(x) \Rightarrow white(x) \vee black(x)$
- "There exists a black swan."
  - $\exists x : swan(x) \wedge black(x)$.
- "A swan is white, unless it is black."
  - $\forall x : swan(x) \wedge \neg black(x) \Rightarrow white(x)$
  - $\forall x : swan(x) \wedge \neg white(x) \Rightarrow black(x)$
  - $\forall x : swan(x) \Rightarrow white(x) \vee black(x)$
- "Not everything that is white or black is a swan."
  - $\neg\forall x : white(x) \vee black(x) \Rightarrow swan(x)$.
  - $\exists x : (white(x) \vee black(x)) \wedge \neg swan(x)$.
- "Black swans have at least one black parent".
  - $\forall x : swan(x) \wedge black(x) \Rightarrow \exists y : swan(y) \wedge black(y) \wedge parent(y, x)$

It is important to recognize the logical structure of an informal sentence in its various equivalent forms.



## The Usage of Formulas

Precise formulation of statements describing object relationships.

- Statement:

  If $x$ and $y$ are natural numbers and $y$ is not zero, then $q$ is the truncated quotient of $x$ divided by $y$.
- Formula:

  $$x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0 \Rightarrow$$
  $$q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$$
- Problem specification:

  Given natural numbers $x$ and $y$ such that $y$ is not zero, compute the truncated quotient $q$ of $x$ divided by $y$.

  - Inputs: $x, y$
  - Input condition: $x \in \mathbb{N} \wedge y \in \mathbb{N} \wedge y \neq 0$
  - Output: $q$
  - Output condition: $q \in \mathbb{N} \wedge \exists r \in \mathbb{N} : r < y \wedge x = y \cdot q + r$



## Problem Specifications

- The specification of a computation problem:
  - Input: variables $x_1 \in S_1, \ldots, x_n \in S_n$
  - Input condition: formula $I(x_1, \ldots, x_n)$.
  - Output: variables $y_1 \in T_1, \ldots, y_m \in T_n$
  - Output condition: formula $O(x_1, \ldots, x_n, y_1, \ldots, y_m)$.
    - $F(x_1, \ldots, x_n)$: only $x_1, \ldots, x_n$ are free in $F$.
    - $x$ is *free* in $F$, if not every occurrence of $x$ is inside the scope of a quantifier (such as $\forall$ or $\exists$) that binds $x$.
- An implementation of the specification:
  - A function (program) $f : S_1 \times \ldots \times S_n \to T_1 \times \ldots \times T_m$ such that

    $$\forall x_1 \in S_1, \ldots, x_n \in S_n : I(x_1, \ldots, x_n) \Rightarrow$$
    $$\textbf{let } (y_1, \ldots, y_m) = f(x_1, \ldots, x_n) \textbf{ in}$$
    $$O(x_1, \ldots, x_n, y_1, \ldots, y_m)$$
  - For all arguments that satisfy the input condition, $f$ must compute results that satisfy the output condition.

Basis of all specification formalisms.

## Example: A Problem Specification

Given an integer array $a$, a position $p$ in $a$, and a length $l$, return the array $b$ derived from $a$ by removing $a[p], \ldots, a[p+l]$.

- Input: $a \in \mathbb{Z}^*$, $p \in \mathbb{N}$, $l \in \mathbb{N}$
- Input condition:

  $p + l \leq \text{length}_{\mathbb{Z}}(a)$
- Output: $b \in \mathbb{Z}^*$
- Output condition:

  **let** $n = \text{length}_{\mathbb{Z}}(a)$ **in**
  $\text{length}_{\mathbb{Z}}(b) = n - l \wedge$
  $(\forall i \in \mathbb{N} : i < p \Rightarrow b[i] = a[i]) \wedge$
  $(\forall i \in \mathbb{N} : p \leq i < n - l \Rightarrow b[i] = a[i+l])$

Mathematical theory:

$T^* := \bigcup_{i \in \mathbb{N}} T^i,\ T^i := \mathbb{N}_i \rightarrow T, \mathbb{N}_i := \{n \in \mathbb{N} : n < i\}$
$\text{length}_T : T^* \rightarrow \mathbb{N}, \text{length}_T(a) = \textbf{such } i \in \mathbb{N} : a \in T^i$



## Validating Problem Specifications

Given a problem specification with input condition $I(x)$ and output condition $O(x, y)$.

- Correctness: take some legal input(s) $a$ with legal output(s) $b$.
  - Check that $I(a)$ and $O(a, b)$ indeed hold.
- Falseness: take some legal input(s) $a$ with illegal output(s) $b$.
  - Check that $I(a)$ holds and $O(a, b)$ does not hold.
- Satisfiability: every legal input should have some legal output.
  - Check $\forall x : I(x) \Rightarrow \exists y : O(x, y)$.
- Non-triviality: for every legal input not every output should be legal.
  - Check $\forall x : I(x) \Rightarrow \exists y : \neg O(x, y)$.

A formal specification does not necessarily capture our intention!



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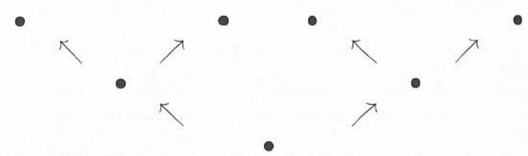
## Proofs

A proof is a structured argument that a formula is true.

- A tree whose nodes represent proof situations (states).



- Each proof situation consists of knowledge and a goal.
  - $K_1, \ldots, K_n \vdash G$
  - Knowledge $K_1, \ldots, K_n$: formulas assumed to be true.
  - Goal $G$: formula to be proved relative to knowledge.
- The root of the tree is the initial proof situation.
  - $K_1, \ldots, K_n$: axioms of mathematical background theories.
  - $G$: formula to be proved.

## Proof Rules

A proof rules describes how a proof situation can be reduced to zero, one, or more "subsituations".

$$\frac{\ldots \vdash \ldots \qquad \ldots \vdash \ldots}{K_1, \ldots, K_n \vdash G}$$

- Rule may or may not close the (sub)proof:
  - Zero subsituations: $G$ has been proved, (sub)proof is closed.
  - One or more subsituations: $G$ is proved, if all subgoals are proved.
- Top-down rules: focus on $G$.
  - $G$ is decomposed into simpler goals $G_1, G_2, \ldots$
- Bottom-up rules: focus on $K_1, \ldots, K_n$.
  - Knowledge is extended to $K_1, \ldots, K_n, K_{n+1}$.

In each proof situation, we aim at showing that the goal is "apparently" true with respect to the given knowledge.



## Conjunction $F_1 \wedge F_2$

$$\frac{K \vdash G_1 \qquad K \vdash G_2}{K \vdash G_1 \wedge G_2} \qquad \frac{\ldots, K_1 \wedge K_2, K_1, K_2 \vdash G}{\ldots, K_1 \wedge K_2 \vdash G}$$

- Goal $G_1 \wedge G_2$.
  - Create two subsituations with goals $G_1$ and $G_2$.

    *We have to show $G_1 \wedge G_2$.*
    - *We show $G_1$: …(proof continues with goal $G_1$)*
    - *We show $G_2$: …(proof continues with goal $G_2$)*
- Knowledge $K_1 \wedge K_2$.
  - Create one subsituation with $K_1$ and $K_2$ in knowledge.

    *We know $K_1 \wedge K_2$. We thus also know $K_1$ and $K_2$. (proof continues with current goal and additional knowledge $K_1$ and $K_2$)*



## Disjunction $F_1 \vee F_2$

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2} \qquad \frac{\ldots, K_1 \vdash G \qquad \ldots, K_2 \vdash G}{\ldots, K_1 \vee K_2 \vdash G}$$

- Goal $G_1 \vee G_2$.
  - Create one subsituation where $G_2$ is proved under the assumption that $G_1$ does not hold (or vice versa):

    *We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show $G_2$. (proof continues with goal $G_2$ and additional knowledge $\neg G_1$)*
- Knowledge $K_1 \vee K_2$.
  - Create two subsituations, one with $K_1$ and one with $K_2$ in knowledge.

    *We know $K_1 \vee K_2$. We thus proceed by case distinction:*
    - *Case $K_1$: …(proof continues with current goal and additional knowledge $K_1$).*
    - *Case $K_2$: …(proof continues with current goal and additional knowledge $K_2$).*



## Implication $F_1 \Rightarrow F_2$

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad \frac{\ldots \vdash K_1 \qquad \ldots, K_2 \vdash G}{\ldots, K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
  - Create one subsituation where $G_2$ is proved under the assumption that $G_1$ holds:

    *We have to show $G_1 \Rightarrow G_2$. We assume $G_1$ and show $G_2$. (proof continues with goal $G_2$ and additional knowledge $G_1$)*
- Knowledge $K_1 \Rightarrow K_2$
  - Create two subsituations, one with goal $K_1$ and one with knowledge $K_2$.

    *We know $K_1 \Rightarrow K_2$.*
    - *We show $K_1$: …(proof continues with goal $K_1$)*
    - *We know $K_2$: …(proof continues with current goal and additional knowledge $K_2$).*

## Equivalence $F_1 \Leftrightarrow F_2$

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots, (\neg)K_2 \vdash G}{\ldots, K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    *We have to show $G_1 \Leftrightarrow G_2$.*
    - *We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$)*
    - *We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)*
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$ .

    *We know $K_1 \Leftrightarrow K_2$.*
    - *We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$)*
    - *We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$)*



## Universal Quantification $\forall x : F$

$$\frac{K \vdash G[x_0/x]}{K \vdash \forall x : G} \; (x_0 \text{ new for } K, G) \qquad \frac{\ldots, \forall x : K, K[T/x] \vdash G}{\ldots, \forall x : K \vdash G}$$

- Goal $\forall x : G$
  - Introduce new (arbitrarily named) constant $x_0$ and create one subsituation with goal $G[x_0/x]$.

    *We have to show $\forall x : G$. Take arbitrary $x_0$.*
    *We show $G[x_0/x]$. (proof continues with goal $G[x_0/x]$)*
- Knowledge $\forall x : K$
  - Choose term $T$ to create one subsituation with formula $K[T/x]$ added to the knowledge.

    *We know $\forall x : K$ and thus also $K[T/x]$.*
    *(proof continues with current goal and additional knowledge $K[T/x]$)*



## Existential Quantification $\exists x : F$

$$\frac{K \vdash G[T/x]}{K \vdash \exists x : G} \qquad \frac{\ldots, K[x_0/x] \vdash G}{\ldots, \exists x : K \vdash G} \; (x_0 \text{ new for } K, G)$$

- Goal $\exists x : G$
  - Choose term $T$ to create one subsituation with goal $G[T/x]$.

    *We have to show $\exists x : G$. It suffices to show $G[T/x]$.*
    *(proof continues with goal $G[T/x]$)*
- Knowledge $\exists x : K$
  - Introduce new (arbitrarily named constant) $x_0$ and create one subsituation with additional knowledge $K[x_0/x]$.

    *We know $\exists x : K$. Let $x_0$ be such that $K[x_0/x]$.*
    *(proof continues with current goal and additional knowledge $K[x_0/x]$)*



## Example

We show

(a) $(\exists x : \forall y : P(x, y)) \Rightarrow (\forall y : \exists x : P(x, y))$

We assume

(1) $\exists x : \forall y : P(x, y)$

and show

(b) $\forall y : \exists x : P(x, y)$

Take arbitrary $y_0$. We show

(c) $\exists x : P(x, y_0)$

From (1) we know for some $x_0$

(2) $\forall y : P(x_0, y)$

From (2) we know

(3) $P(x_0, y_0)$

From (3), we know (c). QED

## Example

We show

(a) $(\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x,y)) \Rightarrow (\exists x, y : q(x,y))$

We assume

(1) $(\exists x : p(x)) \wedge (\forall x : p(x) \Rightarrow \exists y : q(x,y))$

and show

(b) $\exists x, y : q(x,y)$

From (1), we know

(2) $\exists x : p(x)$
(3) $\forall x : p(x) \Rightarrow \exists y : q(x,y)$

From (2) we know for some $x_0$

(4) $p(x_0)$

...



## Example (Contd)

...

From (3), we know

(5) $p(x_0) \Rightarrow \exists y : q(x_0, y)$

From (4) and (5), we know

(6) $\exists y : q(x_0, y)$

From (6), we know for some $y_0$

(7) $q(x_0, y_0)$

From (7), we know (b). $\boxed{\text{QED.}}$



## Indirect Proofs

$$\frac{K, \neg G \vdash \text{false}}{K \vdash G} \qquad \frac{K, \neg G \vdash F \quad K, \neg G \vdash \neg F}{K \vdash G} \qquad \frac{\ldots, \neg G \vdash \neg K}{\ldots, K \vdash G}$$

- Add $\neg G$ to the knowledge and show a contradiction.
  - Prove that "false" is true.
  - Prove that a formula $F$ is true and also prove that it is false.
  - Prove that some knowledge $K$ is false, i.e. that $\neg K$ is true.
    - Switches goal $G$ and knowledge $K$ (negating both).

Sometimes simpler than a direct proof.



## Example

We show

(a) $(\exists x : \forall y : P(x,y)) \Rightarrow (\forall y : \exists x : P(x,y))$

We assume

(1) $\exists x : \forall y : P(x,y)$

and show

(b) $\forall y : \exists x : P(x,y)$

We assume

(2) $\neg \forall y : \exists x : P(x,y)$

and show a contradiction.

...

## Example

...

From (2), we know

(3) $\exists y : \forall x : \neg P(x, y)$

Let $y_0$ be such that

(4) $\forall x : \neg P(x, y_0)$

From (1) we know for some $x_0$

(5) $\forall y : P(x_0, y)$

From (5) we know

(6) $P(x_0, y_0)$

From (4), we know

(7) $\neg P(x_0, y_0)$

From (6) and (7), we have a contradiction. QED.



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## The RISC ProofNavigator

- An interactive proving assistant for program verification.
  - Research Institute for Symbolic Computation (RISC), 2005–: http://www.risc.jku.at/research/formal/software/ProofNavigator.
  - Development based on prior experience with PVS (SRI, 1993–).
  - Kernel and GUI implemented in Java.
  - Uses external SMT (satisfiability modulo theories) solver.
    - CVCL (Cooperating Validity Checker Lite) 2.0, CVC3.
  - Runs under Linux (only); freely available as open source (GPL).
- A language for the definition of logical theories.
  - Based on a strongly typed higher-order logic (with subtypes).
  - Introduction of types, constants, functions, predicates.
- Computer support for the construction of proofs.
  - Commands for basic inference rules and combinations of such rules.
  - Applied interactively within a sequent calculus framework.
  - Top-down elaboration of proof trees.

Designed for simplicity of use; applied to non-trivial verifications.



## Using the Software

For survey, see "Program Verification with the RISC ProofNavigator".
For details, see "The RISC ProofNavigator: Tutorial and Manual".

- Develop a theory.
  - Text file with declarations of types, constants, functions, predicates.
  - Axioms (propositions assumed true) and formulas (to be proved).
- Load the theory.
  - File is read; declarations are parsed and type-checked.
  - Type-checking conditions are generated and proved.
- Prove the formulas in the theory.
  - Human-guided top-down elaboration of proof tree.
  - Steps are recorded for later replay of proof.
  - Proof status is recorded as "open" or "completed".
- Modify theory and repeat above steps.
  - Software maintains dependencies of declarations and proofs.
  - Proofs whose dependencies have changed are tagged as "untrusted".

## Starting the Software

- Starting the software:

```
ProofNavigator &       (32 bit machines at RISC)
ProofNavigator64 &     (64 bit machines at RISC)
```

- Command line options:

```
Usage: ProofNavigator [OPTION]... [FILE]
FILE:   name of file to be read on startup.
OPTION: one of the following options:
        -n, --nogui:  use command line interface.
        -c, --context NAME: use subdir NAME to store context.
        --cvcl PATH: PATH refers to executable "cvcl".
        -s, --silent: omit startup message.
        -h, --help:   print this message.
```

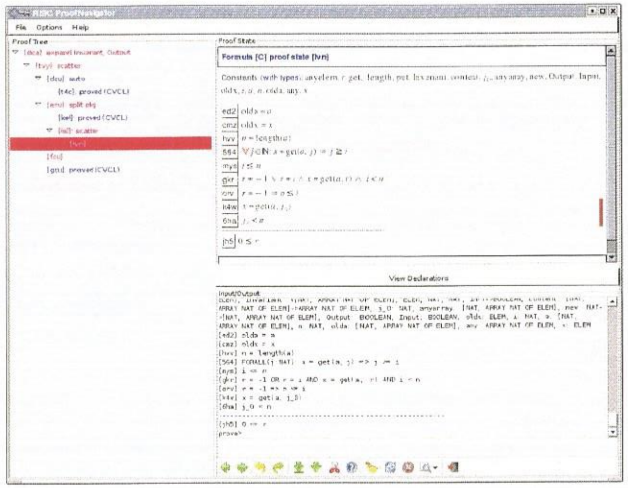
- Repository stored in subdirectory of current working directory:

  `ProofNavigator/`

  - Option `-c` *dir* or command `newcontext "`*dir*`"` :
    - Switches to repository in directory *dir*.



## The Graphical User Interface





## A Theory

```
% switch repository to "sum"
newcontext "sum";

% the recursive definition of the sum from 0 to n
sum: NAT->NAT;
S1: AXIOM sum(0)=0;
S2: AXIOM FORALL(n:NAT): n>0 => sum(n)=n+sum(n-1);

% proof that explicit form is equivalent to recursive definition
S: FORMULA FORALL(n:NAT): sum(n) = (n+1)*n/2;
```

Declarations written with an external editor in a text file.



## Proving a Formula

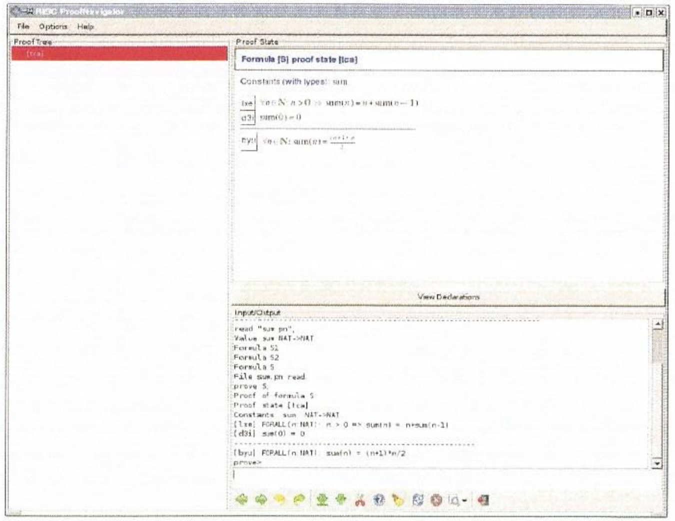
When the file is loaded, the declarations are pretty-printed:

$$\mathrm{sum} \in \mathbb{N} \to \mathbb{N}$$
$$\text{axiom } S1 \equiv \mathrm{sum}(0) = 0$$
$$\text{axiom } S2 \equiv \forall n \in \mathbb{N}: n > 0 \Rightarrow \mathrm{sum}(n) = n + \mathrm{sum}(n-1)$$
$$S \equiv \forall n \in \mathbb{N}: \mathrm{sum}(n) = \frac{(n+1)\cdot n}{2}$$

The proof of a formula is started by the `prove` command.

| Formula S |
| --- |
| prove S: Construct Proof |
| proof S: Show Proof |
| formula S: Print Formula |

# Proving a Formula






# Proving a Formula

- Proof of formula $F$ is represented as a tree.
  - Each tree node denotes a proof state (*goal*).
    - Logical sequent:
      $A_1, A_2, \ldots \vdash B_1, B_2, \ldots$.
    - Interpretation:
      $(A_1 \wedge A_2 \wedge \ldots) \Rightarrow (B_1 \vee B_2 \vee \ldots)$
  - Initially single node $Axioms \vdash F$.

$$\begin{array}{ll} \multicolumn{2}{l}{\text{Constants: } x_0 \in S_0, \ldots} \\ [L_1] & A_1 \\ \ldots & \\ [L_n] & A_n \\ \hline [L_{n+1}] & B_1 \\ \ldots & \\ [L_{n+m}] & B_m \end{array}$$

- The tree must be expanded to completion.
  - Every leaf must denote an obviously valid formula.
    - Some $A_i$ is false or some $B_j$ is true.
- A proof step consists of the application of a proving rule to a goal.
  - Either the goal is recognized as true.
  - Or the goal becomes the parent of a number of children (subgoals).
    The conjunction of the subgoals implies the parent goal.



# An Open Proof Tree

Proof Tree
- [tca]: induction n in byu
  - [dbj]: proved (CVCL)
  - [ebj]

Formula [S] proof state [dbj]

Constants (with types): sum.

lxe | $\forall n \in \mathbb{N}: n > 0 \Rightarrow \mathrm{sum}(n) = n + \mathrm{sum}(n-1)$
d3i | $\mathrm{sum}(0) = 0$

nfq | $\mathrm{sum}(0) = \frac{(0+1)\cdot 0}{2}$

**Parent:** [tca]

Closed goals are indicated in blue; goals that are open (or have open subgoals) are indicated in red. The red bar denotes the "current" goal.



# A Completed Proof Tree

Proof Tree
- [tca]: induction n in byu
  - [dbj]: proved (CVCL)
  - [ebj]: instantiate n_0+1 in lxe
    - [k5f]: proved (CVCL)

The visual representation of the complete proof structure; by clicking on a node, the corresponding proof state is displayed.

## Navigation Commands

Various buttons support navigation in a proof tree.

- : `prev`
  - Go to previous open state in proof tree.
- : `next`
  - Go to next open state in proof tree.
- : `undo`
  - Undo the proof command that was issued in the parent of the current state; this discards the whole proof tree rooted in the parent.
- : `redo`
  - Redo the proof command that was previously issued in the current state but later undone; this restores the discarded proof tree.

Single click on a node in the proof tree displays the corresponding state; double click makes this state the current one.



## Proving Commands

The most important proving commands can be also triggered by buttons.

- (`scatter`)
  - Recursively applies decomposition rules to the current proof state and to all generated child states; attempts to close the generated states by the application of a validity checker.
- (`decompose`)
  - Like `scatter` but generates a single child state only (no branching).
- (`split`)
  - Splits current state into multiple children states by applying rule to current goal formula (or a selected formula).
- (`auto`)
  - Attempts to close current state by instantiation of quantified formulas.
- (`autostar`)
  - Attempts to close current state and its siblings by instantiation.

Automatic decomposition of proofs and closing of proof states.



## Proving Commands

More commands can be selected from the menus.

- `assume`
  - Introduce a new assumption in the current state; generates a sibling state where this assumption has to be proved.
- `case`:
  - Split current state by a formula which is assumed as true in one child state and as false in the other.
- `expand`:
  - Expand the definitions of denoted constants, functions, or predicates.
- `lemma`:
  - Introduce another (previously proved) formula as new knowledge.
- `instantiate`:
  - Instantiate a universal assumption or an existential goal.
- `induction`:
  - Start an induction proof on a goal formula that is universally quantified over the natural numbers.

Here the creativity of the user is required!



## Auxiliary Commands

Some buttons have no command counterparts.

- : `counterexample`
  - Generate a "counterexample" for the current proof state, i.e. an interpretation of the constants that refutes the current goal.
- 
  - Abort current prover activity (proof state simplification or counterexample generation).
- 
  - Show menu that lists all commands and their (optional) arguments.
- 
  - Simplify current state (if automatic simplification is switched off).

More facilities for proof control.

## Proving Strategies

- Initially: semi-automatic proof decomposition.
  - `expand` expands constant, function, and predicate definitions.
  - `scatter` aggressively decomposes a proof into subproofs.
  - `decompose` simplifies a proof state without branching.
  - `induction` for proofs over the natural numbers.
- Later: critical hints given by user.
  - `assume` and `case` cut proof states by conditions.
  - `instantiate` provide specific formula instantiations.
- Finally: simple proof states are yielded that can be automatically closed by the validity checker.
  - `auto` and `autostar` may help to close formulas by the heuristic instantiation of quantified formulas.

Appropriate combination of semi-automatic proof decomposition, critical hints given by the user, and the application of a validity checker is crucial.